# CSE 150A-250A AI: Probabilistic Methods

## Lecture 3

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Slides adapted from previous versions of the course (Prof. Lawrence, Prof. Alvarado, Prof Berg-Kirkpatrick)

# Agenda

# Review

# Review

- **Types of probabilities:**

$$\begin{array}{rl} P(X,Y) & \text{joint} \\ P(Y|X) & \text{conditional} \\ P(X) & \text{unconditional (or marginal)} \end{array}$$

- **Useful rules:**

$$\begin{array}{rcll} P(A,B,C,\ldots) & = & P(A)\,P(B|A)\,P(C|A,B)\ldots & \textbf{product rule} \\ P(X|Y) & = & P(Y|X)P(X)/P(Y) & \textbf{Bayes rule} \\ P(X) & = & \sum_y P(X,Y=y) & \textbf{marginalization} \end{array}$$

- **Conditioning on background evidence $E$:**

$$\begin{array}{rcl} P(A,B,C,\ldots|E) & = & P(A|E)\,P(B|A,E)\,P(C|A,B,E)\ldots \\ P(X|Y,E) & = & P(Y|X,E)P(X|E)/P(Y|E) \\ P(X|E) & = & \sum_y P(X,Y=y|E) \end{array}$$

# Marginal and conditional independence

- Marginal independence

$$\left.\begin{aligned} P(X|Y) &= P(X) \\ P(Y|X) &= P(Y) \\ P(X,Y) &= P(X)\,P(Y) \end{aligned}\right\}$$

*Each of these implies the other two.*

- Conditional independence

$$\left.\begin{aligned} P(X|Y,E) &= P(X|E) \\ P(Y|X,E) &= P(Y|E) \\ P(X,Y|E) &= P(X|E)\,P(Y|E) \end{aligned}\right\}$$

*Each of these also implies the other two.*

# Example of conditional dependence

- $B$ and $E$ are marginally independent:

$$\begin{aligned} P(B) &= P(B|E) \\ P(E) &= P(E|B) \\ P(B,E) &= P(B)\,P(E) \end{aligned}$$

- But $B$ and $E$ are conditionally dependent given $A$:

$$\begin{aligned} P(B|A) &\neq P(B|E,A) \\ P(E|A) &\neq P(E|B,A) \\ P(B,E|A) &\neq P(B|A)\,P(E|A) \end{aligned}$$

# Joint Distributions and Inference

# Why Joints Matter

The joint distribution $P(X_1, \ldots, X_n)$ is a **complete description of uncertainty**.

From the full joint distribution $P(X_1, \ldots, X_n)$, what kinds of probabilities can we compute?

A. Only marginals, $P(X_i)$
B. Only conditionals, $P(X_i|X_j)$
C. Any marginal or conditional over the variables
D. None of the above

**Inference**: Compute the **posterior** distribution of query variables given observed evidence.

- **Model complexity**

Suppose $X_i \in \{0, 1\}$ are binary random variables.

How many numbers do we need to specify the joint distribution of $P(X_1 = x_1, \ldots, X_n = x_n)$?

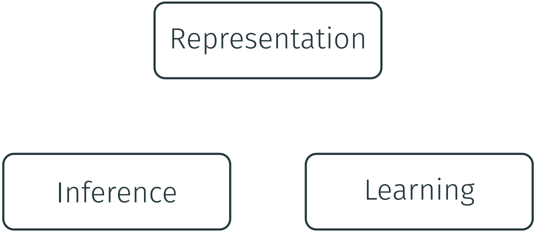
A. $O(n)$

B. $O(2^n)$

C. $O(n^2)$

D. $O(log n)$

It requires numbers to specify the joint distribution $P(X_1 = x_1, \ldots, X_n = x_n)$.




- **Representation**: compactly encode the joint.
- **Inference**: answer queries given evidence.
- **Learning**: estimate structure/parameters from data.

# Alarm Example





- Binary random variables

| | |
|---|---|
| $B \in \{0, 1\}$ | Was there a burglary? |
| $E \in \{0, 1\}$ | Was there an earthquake? |
| $A \in \{0, 1\}$ | Was the alarm triggered? |
| $J \in \{0, 1\}$ | Did Jamal call? |
| $M \in \{0, 1\}$ | Did Maya call? |

# Joint distribution

- Product rule

$$
\begin{aligned}
& P(B, E, A, J, M) \\
& \qquad = \quad P(B)\, P(E|B)\, P(A|B, E)\, P(J|B, E, A)\, P(M|B, E, A, J)
\end{aligned}
$$

*Note: the above is true no matter what the variables signify.*

- Domain-specific assumptions

$$
\begin{aligned}
P(E|B) &= P(E) & \text{marginal independence} \\
P(J|B, E, A) &= P(J|A) & \text{conditional independence} \\
P(M|B, E, A, J) &= P(M|A) & \text{conditional independence}
\end{aligned}
$$

# Completing the model

- Joint distribution

$$
\begin{aligned}
P(B, E, A, J, M) \\
&= P(B)\,P(E|B)\,P(A|B,E)\,P(J|B,E,A)\,P(M|B,E,A,J) \\
&= P(B)\,P(E)\,P(A|B,E)\,P(J|A)\,P(M|A)
\end{aligned}
$$

- Conditional probability tables (CPTs)

| $P(B=1) = 0.001$ |
|---|

| $P(E=1) = 0.002$ |
|---|

| $B$ | $E$ | $P(A=1 \mid B,E)$ |
|---|---|---|
| 0 | 0 | 0.001 |
| 1 | 0 | 0.94 |
| 0 | 1 | 0.29 |
| 1 | 1 | 0.95 |

| $A$ | $P(J=1 \mid A)$ |
|---|---|
| 0 | 0.05 |
| 1 | 0.9 |

| $A$ | $P(M=1 \mid A)$ |
|---|---|
| 0 | 0.01 |
| 1 | 0.7 |

- Joint probabilities are easy to compute:

$$
\begin{aligned}
&P(B{=}1, E{=}0, A{=}1, J{=}1, M{=}1) \\
&\quad = \quad P(B{=}1)\, P(E{=}0)\, P(A{=}1|B{=}1, E{=}0)\, P(J{=}1|A{=}1)\, P(M{=}1|A{=}1) \\
&\quad = \quad (0.001)(1-0.002)(0.94)(0.9)(0.7)
\end{aligned}
$$

- Any inference can be expressed in terms of joint probabilities:

$$
\begin{aligned}
&P(B{=}1, E{=}0|M{=}1) \\
&\quad = \quad \frac{P(B{=}1, E{=}0, M{=}1)}{P(M{=}1)} \qquad \boxed{\text{product rule}} \\
&\quad = \quad \frac{\sum_{a,j} P(B{=}1, E{=}0, A{=}a, J{=}j, M{=}1)}{\sum_{b',e',a',j'} P(B{=}b', E{=}e', A{=}a', J{=}j', M{=}1)} \qquad \boxed{\text{marginalization}}
\end{aligned}
$$

But this approach can be very inefficient!

# Efficient inference

How to perform inference most efficiently?

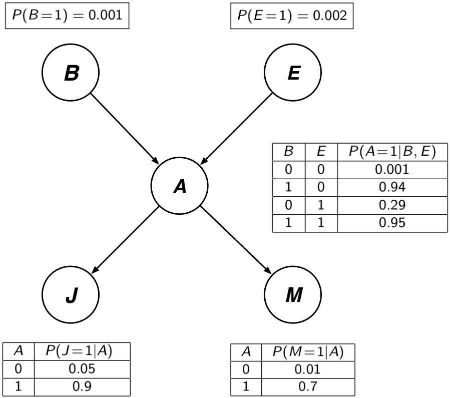
1. Visualize models as directed acyclic graphs.
2. Exploit graph structure to organize and simplify calculations.

*We'll spend today on (1) and next lecture on (2).*

# Visualizing the model

- Joint distribution

$$
\begin{aligned}
P(B, E, A, J, M) \\
&= P(B)\,P(E|B)\,P(A|B,E)\,P(J|B,E,A)\,P(M|B,E,A,J) \\
&= P(B)\,P(E)\,P(A|B,E)\,P(J|A)\,P(M|A)
\end{aligned}
$$

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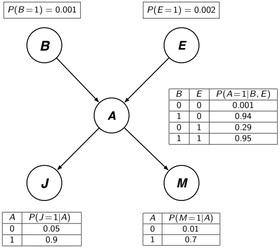
This visual representation of the joint distribution is called a **belief network** (or a **Bayesian network**, or a **probabilistic graphical model**).

# Belief networks

A **belief network** (BN) is a directed acyclic graph (DAG) in which:

1. Nodes represent random variables.
2. Edges represent dependencies.
3. Conditional probability tables (CPTs) describe how each node depends on its parents.

BN = DAG + CPTs




| $B$ | $E$ | $P(A=1 \mid B, E)$ |
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# From distributions to graphs

- It is always true from the product rule that

$$
\begin{aligned}
P(X_1, X_2, \ldots, X_n) &= P(X_1)\, P(X_2|X_1)\, \ldots\, P(X_n|X_1, \ldots, X_{n-1}) \\
&= \prod_{i=1}^{n} P(X_i|X_1, X_2, \ldots, X_{i-1})
\end{aligned}
$$

- But suppose in a particular domain that

$$P(X_i|X_1, X_2, \ldots, X_{i-1}) = P(X_i|\mathrm{parents}(X_i)),$$

where $\mathrm{parents}(X_i)$ is a subset of $\{X_1, \ldots, X_{i-1}\}$.

- Big idea: represent conditional dependencies by a DAG.

# Constructing a belief network

Three steps:

1. Choose your random variables of interest.
2. Choose an ordering of these variables (e.g., $X_1, X_2, \ldots, X_n$).
3. While there are variables left:
   (a) add the node $X_i$ to the network
   (b) set the parents of $X_i$ to be the minimal subset satisfying

$$P(X_1, X_2, \ldots, X_n) \;=\; \prod_{i=1}^{n} P(X_i|\mathrm{parents}(X_i)),$$

   (c) define the conditional probability table $P(X_i|\mathrm{parents}(X_i))$

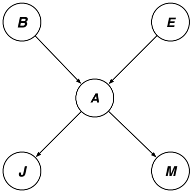
# Node ordering

- Best ordering:

  Add the “root causes,” then the variables they influence, then the next variables that are influenced, etc.

- Example:

  In the alarm world, a natural ordering is $(B, E, A, J, M)$.



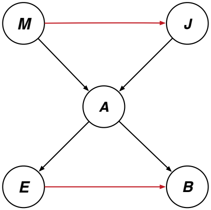
# Node ordering

- What happens if we choose an unnatural ordering?

Ex: $(M, J, A, E, B)$

- Adding nodes with this ordering:

$$
\begin{aligned}
& P(M, J, A, E, B) \\
& \quad = \quad P(M)\, P(J|M)\, P(A|J, M)\, P(E|A, J, M)\, P(B|E, A, J, M) \\
& \quad = \quad P(M)\, P(J|M)\, P(A|J, M)\, P(E|A)\, P(B|A, E)
\end{aligned}
$$




This belief network has two extra edges.
This DAG does not show $P(B) = P(B|E)$.
This DAG does not show $P(M|A) = P(M|A, J)$.
This belief network has larger CPTs.
These CPTs may be more difficult to assess.

# Advantages of belief networks

1. **Compact representation of complex models**

   BNs provide a complete but parsimonious representation of joint probability distributions.

2. **Crisp separation of qualitative vs quantitative knowledge**

| | |
|---|---|
| Qualitative | DAGs encode assumptions of marginal and conditional independence. |
| Quantitative | CPTs encode numerical influences of some variables on others. |

That’s all folks!